

HAMILTON-JACOBI TREATMENT OF CONSTRAINT FIELD SYSTEMS

Walaa I. Eshraim
New York University Abu Dhabi
Saadiyat Island, P.O. Box 129188
Abu Dhabi, U.A.E.

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Abstract

Motivated by the Hamilton–Jacobi approach of fields with constraints, we analyse the classical structure of three different constrained fields systems: (i) the scalar field coupled to two flavours of fermions through Yukawa couplings (ii) the scalar field coupled minimally to the vector potential (iii) the electromagnetic field coupled to a spinor. The equations of motion are obtained as total differential equations in many variables. The integrability conditions are investigated. The second and third constrained systems are quantized using canonical path integral formulation based on the Hamilton–Jacobi treatment.

keywords: Hamiltonian–Jacobi formalism, constrained systems, path integral.

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1 Introduction

The most common method for investigating the Hamiltonian treatment of constrained systems was initiated by Dirac [1,2]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Besides, he showed that the number of degrees of freedom of the dynamical system can be reduced. Hence, the equations of motion of a constrained system are obtained in terms of arbitrary parameters. Moreover, the Dirac approach is widely used for quantizing the constrained Hamilton systems. The path integral is another approach used for the quantization of constrained systems of classical singular theories, which was initiated by Faddeev [3]. Faddeev has applied this approach when only first-class constraints in the canonical gauge are present. Senjanovic [4] generalized Faddeev's method to second-class constraints. Fradkin and Vilkovisky [5,6] red-rived both results in a broader context, where they improved the procedure to the Grassman variables. Gitman and Tyutin [7] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge.

The canonical method (or Güler's method) developed Hamilton-Jacobi formulation to investigate constrained systems [8-9]. The starting point of the Hamilton-Jacobi approach [10-14] is the variational principle. The Hamiltonian treatment of constrained systems leads us to the equations of motion as total differential equations in many variables. The equations are integrable if the corresponding system of partial differential equations is a Jacobi system. In Ref. [15] Güler has presented a treatment of classical fields as constrained systems. Then Hamilton-Jacobi quantization of finite dimensional system with constraints was investigated in Ref. [16]. The advantages of the Hamilton-Jacobi formalism [17-21] are that there are no differences between first- and second-class constraints and no need for a gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations. The Hamilton-Jacobi approach treats the constrained system as the same as Dirac's methods but in a simple way. Both methods give the same results as seen in Refs. [22-26].

This work is organized as follows: In Sec.2 Hamilton-Jacobi formula-

tion is presented. In Sec.3 the Hamilton-Jacobi formulation of the scalar field coupled to two flavours of fermions through Yukawa couplings is investigated. In Sec.4 the path integral quantization of the scalar field coupled minimally to the vector potential is obtained. In Sec.5 the system as the electromagnetic field coupled to a spinor is quantized using Hamilton-Jacobi formulation. In Sec.4 The conclusions are given.

2 Hamilton-Jacobi Formulation

One starts from singular Lagrangian $L \equiv L(q_i, \dot{q}_i, t)$, $i = 1, 2, \dots, n$, with the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (1)$$

of rank $(n - r)$, $r < n$. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \mu = n - r + 1, \dots, n. \quad (3)$$

where q_i are divided into two sets, q_a and x_μ . Since the rank of the Hessian matrix is $(n - r)$, one may solve Eq. (4) for \dot{q}_a as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{x}_\mu, p_a; t). \quad (4)$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t). \quad (5)$$

The canonical Hamiltonian H_0 reads

$$H_0 = -L(q_i, \dot{x}_\nu, \dot{q}_a; t) + p_a \dot{q}_a - \dot{x}_\mu H_\mu, \quad \nu = 1, 2, \dots, r. \quad (6)$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha \left(x_\beta, q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial x_\beta} \right) = 0, \quad \alpha, \beta = 0, 1, \dots, r, \quad (7)$$

where

$$H'_0 = p_0 + H_0, \quad (8)$$

$$H'_\mu = p_\mu + H_\mu. \quad (9)$$

We define $p_\beta = \partial S[q_a; x_a]/\partial x_\beta$ and $p_a = \partial S[q_a; x_a]/\partial q_a$ with $x_0 = t$ and S being the action.

Now the total differential equations are given as

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dx_\alpha, \quad (10)$$

$$dp_a = \frac{\partial H'_\alpha}{\partial q_a} dx_\alpha, \quad (11)$$

$$dp_\beta = \frac{\partial H'_\alpha}{\partial t_\beta} dx_\alpha, \quad (12)$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dx_\alpha, \quad (13)$$

where $Z = S(x_\alpha, q_a)$. These equations are integrable if and only if [27]

$$dH'_0 = 0, \quad (14)$$

$$dH'_\mu = 0, \quad \mu = 1, 2, \dots, r. \quad (15)$$

If conditions (14), (15) are not satisfied identically, one considers them as a new constants and a gain consider their variations. Thus, repeating this procedure, one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the set of canonical phase space coordinates (q_a, p_a) as functions of t_a ; the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters t_α , respectively. In this case the path integral representation can be written as [28, 29].

$$\langle Out | S | In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a \exp \left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \quad (16)$$

$$a = 1, \dots, n-p, \quad \alpha = 0, n-p+1, \dots, n.$$

In fact, this path integral is an integration over the canonical phase space coordinates (q^a, p^a) .

3 Hamilton-Jacobi formulation of the scalar field coupled to two flavours of fermions through Yukawa couplings

We consider one loop order the self-energy for the scalar field φ with a mass m , coupled to two flavours of fermions with masses m_1 and m_2 , coupled through Yukawa couplings described by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 \varphi^2 - \frac{1}{6}\lambda \varphi^3 + \sum_i \bar{\psi}_{(i)}(i\gamma^\mu \partial_\mu - m_i)\psi_{(i)} - g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad \mu = 0, 1, 2, 3, \quad (17)$$

where λ is parameter and g constant, φ , $\psi_{(i)}$, and $\bar{\psi}_{(i)}$ are odd ones. We are adopting the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

The Lagrangian function (17) is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (18)$$

is one.

The generalized momenta (2,3)

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \partial^0 \varphi, \quad (19)$$

$$p_{(i)} = \frac{\partial L}{\partial \dot{\psi}_{(i)}} = i\bar{\psi}_{(i)}\gamma^0 = -H_{(i)}, \quad i = 1, 2, \quad (20)$$

$$\bar{p}_{(i)} = \frac{\partial L}{\partial \dot{\bar{\psi}}_{(i)}} = 0 = -\bar{H}_{(i)}. \quad (21)$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual $\psi_{(i)}$ as a column vector and $\bar{\psi}_{(i)}$ as a row vector implies that $p_{(i)}$ will be a row vector while $\bar{p}_{(i)}$ will be a column vector.

Since the rank of the Hess matrix is one, one may solve (19) for $\partial^0 \varphi$ as:

$$\partial^0 \varphi = p_\varphi \equiv \omega. \quad (22)$$

The usual Hamiltonian H_0 is given as:

$$H_0 = -L + \omega p_\varphi + \partial_0 \psi_{(i)} p_{(i)} \Big|_{p_{(i)} = -H_{(i)}} + \partial_0 \bar{\psi}_{(i)} \bar{p}_{(i)} \Big|_{\bar{p}_{(i)} = -\bar{H}_{(i)}}, \quad (23)$$

or

$$H_0 = \frac{1}{2}(p_\varphi^2 - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad a = 1, 2, 3. \quad (24)$$

Eqs. (20), and (21) lead to the primary constraints.

By using Hamilton-Jacobi, the set of (HJPDE) (8) read as

$$H'_0 = p_0 + H_0 = p_0 + \frac{1}{2}(p_\varphi^2 - \partial_a \varphi \partial^a \varphi) + \frac{1}{2}m^2 \varphi^2 + \frac{1}{6}\lambda \varphi^3 - \bar{\psi}_{(i)}(i\gamma^a \partial_a - m_i)\psi_{(i)} + g\varphi(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}), \quad (25)$$

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\bar{\psi}_{(i)}\gamma^0 = 0, \quad (26)$$

$$\bar{H}'_{(i)} = \bar{p}_{(i)} + \bar{H}_{(i)} = \bar{p}_{(i)} = 0. \quad (27)$$

Therefore, the total differential equations for the characteristic (10), (11) and (12) are:

$$d\varphi = p_\varphi d\tau, \quad (28)$$

$$d\psi_{(i)} = d\psi_{(i)}, \quad (29)$$

$$d\bar{\psi}_{(i)} = d\bar{\psi}_{(i)}, \quad (30)$$

$$dp_\varphi = \left[m^2 \varphi + \frac{1}{2}\lambda \varphi^2 + g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) \right] d\tau, \quad (31)$$

$$dp_{(1)} = \left[\bar{\psi}_{(1)}(i\overleftarrow{\partial}_a \gamma^a + m_1) + g\varphi \bar{\psi}_{(2)} \right] d\tau, \quad (32)$$

$$dp_{(2)} = \left[\bar{\psi}_{(2)} (i \overleftarrow{\partial}_a \gamma^a + m_2) + g \varphi \bar{\psi}_{(1)} \right] d\tau, \quad (33)$$

$$d\bar{p}_{(1)} = \left[- (i \gamma^a \partial_a - m_1) \psi_{(1)} + g \varphi \psi_{(2)} \right] d\tau - i \gamma^0 d\psi_{(1)}, \quad (34)$$

$$d\bar{p}_{(2)} = \left[- (i \gamma^a \partial_a - m_2) \psi_{(2)} + g \varphi \psi_{(1)} \right] d\tau - i \gamma^0 d\psi_{(2)}. \quad (35)$$

The integrability conditions ($dH'_\alpha = 0$) imply that the variation of the constraints $H'_{(i)}$ and $\bar{H}'_{(i)}$ should be identically zero, that is

$$dH'_{(i)} = dp_{(i)} - i d\bar{\psi}_{(i)} \gamma^0 = 0, \quad (36)$$

$$d\bar{H}'_{(i)} = d\bar{p}_{(i)} = 0. \quad (37)$$

The following equations of motion:
From Eq. (28), we obtain

$$\dot{\varphi} = p_\varphi. \quad (38)$$

Substituting from Eqs. (32) and (33) into Eq. (36), we get

$$i \partial_0 \bar{\psi}_{(1)} \gamma^0 - \bar{\psi}_{(1)} (i \overleftarrow{\partial}_a \gamma^a + m_1) - g \varphi \bar{\psi}_{(2)} = 0, \quad (39)$$

$$i \partial_0 \bar{\psi}_{(2)} \gamma^0 - \bar{\psi}_{(2)} (i \overleftarrow{\partial}_a \gamma^a + m_2) - g \varphi \bar{\psi}_{(1)} = 0. \quad (40)$$

Substituting from Eqs. (34) and (35) into Eq. (37), we have

$$(i \gamma^\mu \partial_\mu - m_1) \psi_{(1)} - g \varphi \psi_{(2)} = 0, \quad (41)$$

$$(i \gamma^\mu \partial_\mu - m_2) \psi_{(2)} - g \varphi \psi_{(1)} = 0. \quad (42)$$

One notes that the integrability conditions are not identically zero, they are added to the set of equations of motion.

From Eqs.(31-33), we get the following equations of motion:

$$\dot{p}_\varphi = m^2 \varphi + \frac{1}{2} \lambda \varphi^2 + g (\bar{\psi}_{(1)} \psi_{(2)} + \bar{\psi}_{(2)} \psi_{(1)}), \quad (43)$$

$$\dot{p}_{(1)} = \bar{\psi}_{(1)}(i\overleftarrow{\partial}_a\gamma^a + m_1) + g\varphi\bar{\psi}_{(2)}, \quad (44)$$

$$\dot{p}_{(2)} = \bar{\psi}_{(2)}(i\overleftarrow{\partial}_a\gamma^a + m_2) + g\varphi\bar{\psi}_{(1)}. \quad (45)$$

Substituting from Eqs. (41) and (42) into Eqs. (34) and (35), we get

$$\dot{\vec{p}}_{(i)} = 0, \quad i = 1, 2. \quad (46)$$

Differentiate Eq. (38) with respect to time, and making use of Eq. (43), we get

$$\ddot{\varphi} - m^2\varphi - \frac{1}{2}\lambda\varphi^2 - g(\bar{\psi}_{(1)}\psi_{(2)} + \bar{\psi}_{(2)}\psi_{(1)}) = 0. \quad (47)$$

4 Path integral quantization of the scalar field coupled minimally to the vector potential

Consider the action integral for the scalar field coupled minimally to the vector potential as

$$S = \int d_4x L, \quad (48)$$

where the Lagrangian L is given by

$$L = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + (D_\mu\varphi)^*(x)D^\mu\varphi(x) - m^2\varphi^*(x)\varphi(x), \quad (49)$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (50)$$

and

$$D_\mu\varphi(x) = \partial_\mu\varphi(x) - ieA_\mu(x)\varphi(x). \quad (51)$$

The canonical momenta are defined as

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \quad (52)$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \quad (53)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (D_0 \varphi)^* = \dot{\varphi}^* + ie A_0 \varphi^*, \quad (54)$$

$$p_{\varphi^*} = \frac{\partial L}{\partial \dot{\varphi}^*} = (D_0 \varphi) = \dot{\varphi} - ie A_0 \varphi, \quad (55)$$

From Eqs. (52), (54) and (55), the velocities \dot{A}_i , $\dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i , p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \quad (56)$$

$$\dot{\varphi}^* = p_\varphi - ie A_0 \varphi^*, \quad (57)$$

$$\dot{\varphi} = p_{\varphi^*} + ie A_0 \varphi. \quad (58)$$

The canonical Hamiltonian H_0 is obtained as

$$H_0 = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + p_{\varphi^*} p_\varphi + ie A_0 \varphi p_\varphi - ie A_0 \varphi^* p_{\varphi^*} - (D_i \varphi)^* (D^i \varphi) + m^2 \varphi^* \varphi. \quad (59)$$

Making use of Eqs. (7) and (9), we find for the set of HJPDE

$$H'_0 = \pi_4 + H_0, \quad (60)$$

$$H' = \pi_0 + H = \pi_0 = 0, \quad (61)$$

Therefore, the total differential equations for the characteristic (10-12) obtained as

$$\begin{aligned} dA^i &= \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'}{\partial \pi_i} dA^0, \\ &= -(\pi^i + \partial_i A_0) dt, \end{aligned} \quad (62)$$

$$dA^0 = \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'}{\partial \pi_0} dA^0 = dA^0, \quad (63)$$

$$\begin{aligned} d\varphi &= \frac{\partial H'_0}{\partial p_\varphi} dt + \frac{\partial H'}{\partial p_\varphi} dA^0, \\ &= (p_{\varphi^*} + ieA_0\varphi) dt, \end{aligned} \quad (64)$$

$$\begin{aligned} d\varphi^* &= \frac{\partial H'_0}{\partial p_{\varphi^*}} dt + \frac{\partial H'}{\partial p_{\varphi^*}} dA^0, \\ &= (p_\varphi - ieA_0\varphi^*) dt, \end{aligned} \quad (65)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0, \\ &= [\partial_i F^{li} + ie(\varphi^* \partial^i \varphi + \varphi \partial_i \varphi^*) + 2e^2 A^i \varphi \varphi^*] dt, \end{aligned} \quad (66)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'}{\partial A_0} dA^0, \\ &= [\partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi] dt, \end{aligned} \quad (67)$$

$$\begin{aligned} dp_\varphi &= -\frac{\partial H'_0}{\partial \varphi} dt - \frac{\partial H'}{\partial \varphi} dA^0, \\ &= [(\vec{D} \cdot \vec{D} \varphi)^* - m^2 \varphi^* - ieA_0 p_\varphi] dt, \end{aligned} \quad (68)$$

and

$$\begin{aligned} dp_{\varphi^*} &= -\frac{\partial H'_0}{\partial \varphi^*} dt - \frac{\partial H'}{\partial \varphi^*} dA^0, \\ &= [(\vec{D} \cdot \vec{D} \varphi) - m^2 \varphi + ieA_0 p_{\varphi^*}] dt. \end{aligned} \quad (69)$$

The integrability condition ($dH'_\alpha = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \quad (70)$$

which leads to a new constraint

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi p_\varphi = 0. \quad (71)$$

Taking the total differential of H'' , we have

$$dH'' = \partial_i d\pi^i + i e p_{\varphi^*} d\varphi^* + i e \varphi^* d p_{\varphi^*} - i e \varphi d p_{\varphi} - i e p_{\varphi} d\varphi = 0. \quad (72)$$

Then the set of equation (62-69) is integrable. Therefore, the canonical phase space coordinates (φ, p_{φ}) and $(\varphi^*, p_{\varphi^*})$ are obtained in terms of parameters (t, A^0) .

Making use of Eq. (13) and (59-61), we obtain the canonical action integral as

$$Z = \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + p_{\varphi} p_{\varphi^*} + \vec{D}\varphi^* \cdot \vec{D}\varphi + m^2 |\varphi|^2 \right), \quad (73)$$

where

$$\vec{D} = \vec{\nabla} + i e \vec{A}. \quad (74)$$

Now the path integral representation (16) is given by

$$\begin{aligned} \langle out|S|In \rangle = & \int \prod_i dA^i d\pi^i d\varphi d p_{\varphi} d\varphi^* d p_{\varphi^*} \exp \left[i \left\{ \int d^4x \right. \right. \\ & \left. \left. \left(-\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + p_{\varphi} p_{\varphi^*} + (D_i \varphi)^* (D_i \varphi) - m^2 \varphi^* \varphi \right) \right\} \right]. \quad (75) \end{aligned}$$

5 Path integral quantization of electromagnetic field coupled to a spinor

We analyse the case of the electromagnetic field coupled to a spinor, whose Hamiltonian formalism was analysed [30,31]. We will consider the Lagrangian density written as

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^{\mu} (\partial_{\mu} + i e A_{\mu}) \psi - m \bar{\psi} \psi, \quad (76)$$

where A_{μ} are even variables while ψ and $\bar{\psi}$ are odd ones. The electromagnetic tensor is defined as $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ and we are adopting the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$.

The Lagrangian function (76) is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad (77)$$

is three.

The momenta variables conjugated, respectively, to A_i, A_0, ψ and $\bar{\psi}$, are

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \quad (78)$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0 = -H_1, \quad (79)$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = -H_\psi, \quad (80)$$

$$p_{\bar{\psi}} = \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \quad (81)$$

where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual, ψ as a column vector and $\bar{\psi}$ as a row vector implies that p_ψ will be a row vector while $p_{\bar{\psi}}$ will be a column vector.

With the aid of relation (78), the Lagrangian density may be written as

$$L = -\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F_{ij} F^{ij} + i \bar{\psi} \gamma^\mu (\partial_\mu + ie A_\mu) \psi - m \bar{\psi} \psi, \quad (82)$$

then the canonical Hamiltonian density takes the form

$$H_0 = \pi^i \dot{A}_i + \frac{1}{2} \pi_i \pi^i + \frac{1}{4} F^{ij} F_{ij} - i \bar{\psi} (\gamma^\mu ie A_\mu + \gamma^i \partial_i) \psi + m \bar{\psi} \psi. \quad (83)$$

The velocities \dot{A}_i can be expressed in terms of the momenta π_i as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \quad (84)$$

Therefore, the Hamiltonian density is

$$H_0 = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + \bar{\psi} \gamma^\mu e A_\mu \psi - \bar{\psi} (i \gamma^i \partial_i - m) \psi. \quad (85)$$

The set of Hamilton-Jacobi Partial Differential Equation (HJPDE) reads

$$H'_0 = p_0 + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + \bar{\psi} \gamma^\mu e A_\mu \psi - \bar{\psi} (i \gamma^i \partial_i - m) \psi, \quad (86)$$

$$H'_1 = \pi^0 + H_1 = \pi_0 = 0, \quad (87)$$

$$H'_\psi = p_\psi + H_\psi = p_\psi - i\bar{\psi}\gamma^0 = 0, \quad (88)$$

$$H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0. \quad (89)$$

Therefore, the total differential equations for the characteristic (10), (11) and (12), obtained as

$$\begin{aligned} dA^i &= \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'_1}{\partial \pi_i} dA^0 + \frac{\partial H'_\psi}{\partial \pi_i} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_i} d\bar{\psi}, \\ &= -(\pi^i + \partial_i A_0) dt, \end{aligned} \quad (90)$$

$$\begin{aligned} dA^0 &= \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'_1}{\partial \pi_0} dA^0 + \frac{\partial H'_\psi}{\partial \pi_0} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_0} d\bar{\psi}, \\ &= dA^0, \end{aligned} \quad (91)$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'_1}{\partial A_i} dA^0 - \frac{\partial H'_\psi}{\partial A_i} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_i} d\bar{\psi}, \\ &= (\partial_i F^{hi} - e\bar{\psi}\gamma^i\psi) dt, \end{aligned} \quad (92)$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'_1}{\partial A_0} dA^0 - \frac{\partial H'_\psi}{\partial A_0} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_0} d\bar{\psi}, \\ &= (\partial_i \pi^i - e\bar{\psi}\gamma^0\psi) dt, \end{aligned} \quad (93)$$

$$\begin{aligned} dp_\psi &= -\frac{\partial H'_0}{\partial \psi} dt - \frac{\partial H'_1}{\partial \psi} dA^0 + \frac{\partial H'_\psi}{\partial \psi} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \psi} d\bar{\psi}, \\ &= -(i\gamma^i\partial_i + e\gamma^\mu A_\mu + m)\bar{\psi} dt, \end{aligned} \quad (94)$$

and

$$\begin{aligned} dp_{\bar{\psi}} &= -\frac{\partial H'_0}{\partial \bar{\psi}} dt - \frac{\partial H'_1}{\partial \bar{\psi}} dA^0 + \frac{\partial H'_\psi}{\partial \bar{\psi}} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \bar{\psi}} d\bar{\psi}, \\ &= (-i\gamma^i\partial_i + e\gamma^\mu A_\mu + m)\psi dt - i\gamma^0 d\psi. \end{aligned} \quad (95)$$

The integration condition ($dH'_\alpha = 0$) imply that the variation of the constraints H'_1 , H'_ψ and $H'_{\bar{\psi}}$ should be identically zero

$$dH'_1 = d\pi_0 = 0, \quad (96)$$

$$dH'_\psi = dp_\psi - i\gamma^0 d\bar{\psi} = 0, \quad (97)$$

$$dH'_{\bar{\psi}} = dp_{\bar{\psi}} = 0, \quad (98)$$

when we substituting from Eqs. (94) and (95) into Eqs.(97) and (98) respectively, we obtained as

$$dH'_\psi = 0, \quad (99)$$

and

$$dH'_{\bar{\psi}} = 0, \quad (100)$$

if and only if

$$i\bar{\psi}\gamma^\mu(\overleftarrow{\partial}_\mu - ieA_\mu) + m\bar{\psi} = 0, \quad (101)$$

and

$$i(\partial_\mu + ieA_\mu)\gamma^\mu\psi - m\psi = 0, \quad (102)$$

are satisfied. Then the set of equations (90, 92, 93) are integrable and are just ordinary differential equations and are set in the form

$$\dot{A}^i = -\pi^i - \partial_i A_0, \quad (103)$$

$$\dot{\pi}^i = \partial_i F^{ti} - e\bar{\psi}\gamma^i\psi, \quad (104)$$

$$\dot{\pi}^0 = \partial_i \pi^i - e\bar{\psi}\gamma^0\psi. \quad (105)$$

These are the equations of motions with full gauge freedom. It can be seen, from Eq. (91), that A^0 is an arbitrary (gauge dependent) variable since its time derivative is arbitrary. Besides that, Eq. (103) shows the gauge dependence of A^i and, Taking the curl of its vector form, leads to the known Maxwell equation

$$\frac{\partial \vec{A}}{\partial t} = -\vec{E} - \vec{\nabla}(A_0 - \alpha) \Rightarrow \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}. \quad (106)$$

Writing $j^\mu = e\bar{\psi}\gamma^\mu\psi$ we get, from Eq. (104), the inhomogeneous Maxwell equation

$$\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \vec{j}, \quad (107)$$

while the other inhomogeneous equation

$$\vec{\nabla} \cdot \vec{E} = j^0, \quad (108)$$

follows from Eq. (105). Expressions (101) and (102) are the known equations for the spinor ψ and $\bar{\psi}$.

Eqs. (12) and (86-89) lead us to the canonical action integral as

$$Z = \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i \pi^i + \pi^i \dot{A}_i + \pi^i \partial_i A_0 + i\bar{\psi}\gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi}\psi \right). \quad (109)$$

Making use of equations (14) and (109), we obtained the path integral as

$$\langle out|S|In \rangle = \int \prod_i dA^i d\pi^i d\psi d\bar{\psi} \exp \left[i \left\{ \int d^4x \left(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i \pi^i + \pi^i \dot{A}_i + \pi^i \partial_i A_0 + i\bar{\psi}\gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi}\psi \right) \right\} \right]. \quad (110)$$

Integration over π_i gives

$$\langle out|S|In \rangle = N \int \prod_i dA^i d\psi d\bar{\psi} \exp \left[i \left\{ \int d^4x \left(\frac{1}{2} (\dot{A}^i + \partial_i A_0)^2 - \frac{1}{4} F^{ij} F_{ij} + i\bar{\psi}\gamma^\mu (\partial_\mu + ieA_\mu) \psi - m\bar{\psi}\psi \right) \right\} \right]. \quad (111)$$

6 Conclusion

In this paper three different constrained fields systems are studied by using Hamilton-Jacobi formulation. Firstly, the scalar field coupled to two flavours of fermions through Yukawa couplings is discussed as constrained

system by using Hamilton-Jacobi methods. The equations of motion are obtained without introducing Lagrange multipliers to the canonical Hamiltonian.

Then, we obtained the path integral quantization of the scalar field coupled minimally to the vector potential by using the canonical path integral formulation. The integrability conditions dH'_0 and dH' are satisfied, the system is integrable, hence the path integral is obtained directly as an integration over the canonical phase space coordinates $A^i, \pi^i, \varphi, P_\varphi, \varphi^*$ and p_{φ^*} without using any gauge fixing conditions.

Finally, the path integral quantization of the electromagnetic field coupled to a spinor is also obtained by using the canonical path integral formalism. The integrability conditions dH'_1, dH'_ψ and $dH'_{\bar{\psi}}$ are identically satisfied, and the system is integrable. Hence, the canonical phase space coordinates $(A^i, \pi^i), (\psi, p_\psi)$ and $(\bar{\psi}, p_{\bar{\psi}})$ are obtained in terms of the parameter τ . The path integral is obtained as an integration over the canonical phase-space coordinates (A^i, π^i) and $(\psi, \bar{\psi})$ without using any gauge fixing condition. From the equations of motion for this system, we obtained the inhomogeneous Maxwell equation.

One can see many advantages of this path integral formalism, which are no need to enlarge the initial phase-space by introducing unphysical auxiliary field, no need to distinguish between first and second-class constraints, no need to introduce Lagrange multipliers, no need to use delta functions in the measure as well as to use gauge fixing conditions; all that needed is the set of Hamilton-jacobi partial differential equations and the equations of motions. If the system is integrable, then one can construct the reduced canonical phase-space.

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